

On the stability of one particle states generated by quantum fields fulfilling Yang-Feldman equations

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Abstract

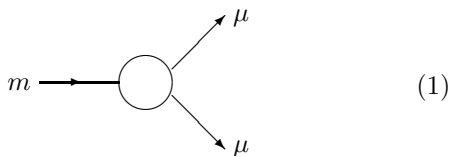
We prove that for a Wightman quantum field $\phi(x)$ the assumptions (i) positivity of the metric on the state space \mathcal{H} of the theory (ii) the asymptotic condition in the form of Yang-Feldman equations and (iii) Klein-Gordon equation for the outgoing field imply that the states generated by application of the asymptotic fields to the vacuum are stable. We prove by a counter example that this statement is wrong in the case of quantum fields with indefinite metric.

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1 Introduction

It is well-known that in positive metric quantum field theory (QFT) the decay reaction of a particle with mass $m > 2\mu$ into two particles of mass μ



is prohibited kinematically, since the one particle in-state has sharp mass m and the two particle out-state has continuous mass spectrum. In this note we prove a generalization of this statement which shows that in

positive metric quantum field theory the asymptotic condition in form of the Yang-Feldman equations[13] and the Klein-Gordon equations for the asymptotic fields are in contradiction with a nontrivial transition of a one particle in-state into an arbitrary state (not necessarily a multi-particle state). No restrictions on the spectrum of this state have to be assumed. This might be of particular interest in QFTs containing several species of particles with the same mass or QFTs which are not asymptotically complete [10].

Positivity crucially enters into the proof of this statement and we show by a counter example that there exist quantum fields with indefinite metric such that even a non-trivial decay reaction (1) is possible.

The article is organized as follows: In the next section we state our assumptions and we give a suitable formulation of Yang-Feldman equations. In section 3 we state and prove the main theorem. Section 4 is devoted to the counter example for the case of QFTs with indefinite metric.

2 Preparations

Let $\phi(x)$ be a relativistic Wightman field (cf. [12]) acting on the Hilbert space \mathcal{H} with vacuum Ψ_0 . The theory is allowed to contain an arbitrary number of quantum fields, in particular we do not assume cyclicity of the vacuum w.r.t. $\phi(x)$. For simplicity we assume $\phi(x)$ to be Hermitean and scalar (these assumptions however are not crucial). $j(x)$ we denote

the current fields of $\phi(x)$, i.e.

$$j(x) = (-\partial_\nu \partial^\nu - m^2)\phi(x). \quad (2)$$

Next we come to the question, how to define asymptotic states for the field $\phi(x)$ without assuming an *isolated* mass-shell of mass m in the spectrum of the energy-momentum operator \mathbf{P} (this automatically would imply stability of one particle states). Haag-Ruelle [4, 5, 11] theory can not be applied in this situation. We help ourselves by *postulating* the asymptotic condition [8] in the following rather weak form based on the Yang-Feldman equations: Let $f \rightarrow \phi^{\text{in/out}}(f)\Psi_0$ be a vector valued distribution which assigns to a test function f in the space of rapidly decreasing test functions $\mathcal{S} = \mathcal{S}(\mathbb{R}^4)$ a vector in the state space \mathcal{H} of the theory. We postulate that the Yang-Feldman equations for ϕ hold for certain matrix elements, in particular

$$\begin{aligned} &\langle \phi(x)\Psi_0, \Psi \rangle = \\ &\left\langle \left[(-\partial_\nu \partial^\nu - m^2)_{\text{ret/adv}}^{-1} * j(x) + \phi^{\text{in/out}}(x) \right] \right. \\ &\left. \times \Psi_0, \Psi \right\rangle \quad \forall \Psi \in \mathcal{H} \end{aligned} \quad (3)$$

(here it would be sufficient to take Ψ from a dense subset of \mathcal{H}). In order to give a precise sense to Eq. (3) we have to assume that the distributional convolution of $j(x)$ with the retarded/advanced Greens function $(-\partial_\nu \partial^\nu - m^2)_{\text{ret/adv}}^{-1}$ is well-defined in (3). A suitable technical formulation is the following: Let $\hat{w}(k) = \hat{w}(k, \Psi)$ be defined as

$$\hat{w}(k) = \int_{\mathbb{R}^4} e^{ix \cdot k} \langle j(x)\Psi_0, \Psi \rangle dx \quad (4)$$

We have to make sure that the product of $\hat{w}(k)$ and the Fourier transform $1/(k^2 - m^2 \mp ik^0\epsilon)$ of $(-\partial_\nu \partial^\nu - m^2)_{\text{ret/adv}}^{-1}$ is well defined. A sufficient and essentially necessary condition for this is that $\hat{w}(k)$ is continuously differentiable in the variable $\kappa = k^0 - \omega$ in an open neighborhood of $\kappa = 0$, $\omega = \sqrt{|\mathbf{k}|^2 + m^2}$. We shall assume this in the following. Furthermore, we assume the Klein-Gordon equation for the outgoing field applied to the vacuum

$$(-\partial_\nu \partial^\nu - m^2)\phi^{\text{out}}(x)\Psi_0 = 0. \quad (5)$$

Using the Yang-Feldman equations it is not difficult to see that this implies the Klein-Gordon equation also for the incoming field applied to the vacuum.

3 The theorem

We now can formulate the main result of this note:

Theorem 1 *In any quantum field theory with positive metric the assumptions of Section 2 concerning the Yang-Feldman equations (3) and the Klein-Gordon equation for the outgoing field (5) imply stability of the one particle states of the incoming and outgoing fields in the sense*

$$\phi^{\text{in}}(x)\Psi_0 = \phi^{\text{out}}(x)\Psi_0. \quad (6)$$

The rest of this section is devoted to the proof of this theorem. We suppose that Eq. (6) does not hold to derive a contradiction. If Eq. (6) does not hold then $\exists \Psi \in \mathcal{H}$ such that

$$\langle \phi^{\text{in}}(x)\Psi_0, \Psi \rangle \neq \langle \phi^{\text{out}}(x)\Psi_0, \Psi \rangle. \quad (7)$$

We set

$$\begin{aligned} \hat{W}^{-/\text{in/out}}(k) &= \int_{\mathbb{R}^4} e^{ix \cdot k} \\ &\times \left\langle \phi^{-/\text{in/out}}(x)\Psi_0, \Psi \right\rangle dx, \end{aligned} \quad (8)$$

where $-$ stands for the local field. We then get for the Yang-Feldman equations (3)

$$\hat{W}(k) = \frac{\hat{w}(k)}{(k^2 - m^2) \mp ik^0\epsilon} + \hat{W}^{\text{in/out}}(k) \quad (9)$$

and further more $\hat{W}^{\text{in}}(k) \neq \hat{W}^{\text{out}}(k)$, cf. Eq. (7).

Step 1) We want to show that by (7) the matrix element for the current j can not vanish identically 'on shell': Thus suppose that $\hat{w}(k) = 0$ whenever $\kappa = 0$. Since $\hat{w}(k)$ is continuously differentiable in a neighborhood of $\kappa = 0$ we get that $\hat{w}(k) = \kappa b(k)$ for some function $b(k)$ which is continuous on a neighborhood of $\kappa = 0$. Using that

$\kappa b(k)/(\kappa + ik^0\epsilon) = \kappa b(k)/(\kappa - ik^0\epsilon) = b(k)$ holds in the sense of tempered distributions, we get

$$\frac{\hat{w}(k)}{(k^2 - m^2) - ik^0\epsilon} = \frac{\hat{w}(k)}{(k^2 - m^2) + ik^0\epsilon} \quad (10)$$

and hence, by Eq. (9), $\hat{W}^{\text{in}}(k) = \hat{W}^{\text{out}}(k)$ in contradiction with Eq.(7). Here we also used that $\hat{w}(k)$ vanishes on the backward mass-shell due to the assumptions on the spectrum of \mathbf{P} ([12]). Thus, there must be a Schwartz test function $g : \mathbb{R}^3 \rightarrow \mathbb{R}$ s.t.

$$\tilde{w}(\kappa) = \int_{\mathbb{R}^3} \frac{\hat{w}(\kappa + \omega, \mathbf{k})}{\kappa + 2\omega} g(\mathbf{k}) d\mathbf{k} \quad (11)$$

is continuous differentiable and not equal to zero in a neighborhood of $\kappa = 0$.

Step 2) Now let $\hat{\varphi} : \mathbb{R} \rightarrow \mathbb{R}$ be an antisymmetric function ($\hat{\varphi}(\kappa) = -\hat{\varphi}(-\kappa)$) s.t. $\hat{\varphi}(\kappa) = \kappa$ for $\kappa \in [-1, 1]$, $2 \geq \hat{\varphi}(\kappa) \geq 0$ for $\kappa \geq 0$ with support in $[-2, 2]$. We set $\hat{\varphi}_n(\kappa) = \hat{\varphi}(n\kappa)/\tilde{w}(\kappa)$ which is a continuously differentiable function for sufficiently large n . We observe that for such n

$$\begin{aligned} & \left| \int_{\mathbb{R}^4} \hat{W}(k) \hat{\varphi}_n(k^0 - \omega) g(\mathbf{k}) dk \right| \\ &= \left| \int_{\mathbb{R}} \hat{\varphi}_n(\kappa) \frac{\tilde{w}(\kappa)}{\kappa + i\epsilon} d\kappa \right| \\ &= \left| \int_{\mathbb{R}} \frac{\hat{\varphi}(n\kappa)}{\kappa} d\kappa \right| \\ &\geq \int_{-1/n}^{1/n} n d\kappa = 2 \end{aligned} \quad (12)$$

In the first step we have used the Yang-Feldman equation in the form (9) for the out-case taking into account $\hat{\varphi}_n(\kappa)\hat{W}^{\text{out}}(k) = 0$ by Eq. (5) and the spectral condition [12]. In the second step we used that $\hat{\varphi}_n(\kappa)$ vanishes for $\kappa = 0$ and we used the definition of $\hat{\varphi}_n(\kappa)$. In the third step we used the fact that by definition of $\hat{\varphi}$ we get $\hat{\varphi}(n\kappa)/\kappa \geq 0$ for all $\kappa \in \mathbb{R}$ and $\hat{\varphi}(n\kappa)/\kappa = n$ for $\kappa \in [-1/n, 1/n]$. The last step is trivial.

Step 3) We define the following sequence of test functions f_n :

$$f_n(x) = (2\pi)^{-4} \int_{\mathbb{R}^4} e^{-ik \cdot x} \hat{\varphi}_n(k^0 - \omega) g(\mathbf{k}) dk. \quad (13)$$

The f_n might not be Schwartz test functions (this happens if $\tilde{w}(\kappa)$ is not infinitely often differentiable in a neighborhood of zero). But evaluation of all distributions we consider will be well-defined for the test functions f_n and the argument we give below can be made fully rigorous by smearing out the Fourier transform of the f_n . We leave the details to the reader. We get by construction

$$2 \leq \lim_{n \rightarrow \infty} |\langle \phi(f_n) \Psi_0, \Psi \rangle|. \quad (14)$$

The positivity of a Hermitean inner product $\langle \cdot, \cdot \rangle$ is equivalent with the Cauchy-Schwarz inequality for this product. Applying this inequality to the above equation we get

$$\begin{aligned} 4 &\leq \langle \Psi, \Psi \rangle \lim_{n \rightarrow \infty} \langle \phi(f_n) \Psi_0, \phi(f_n) \Psi_0 \rangle \\ &= \langle \Psi, \Psi \rangle \lim_{n \rightarrow \infty} \int_0^\infty \left[\int_{\mathbb{R}^4} |\hat{\varphi}_n(k^0 - \omega) g(\mathbf{k})|^2 \right. \\ &\quad \times \left. \delta_s^+(k) dk \right] \rho(ds^2) \end{aligned} \quad (15)$$

where $\delta_s^\pm(k) = \theta(\pm k^0) \delta(k^2 - s^2)$. Here we have used the Källen-Lehmann representation of the two-point function of $\phi(x)$ with ρ the spectral measure of $\phi(x)$. Now, the functions $|\hat{\varphi}_n(k^0 - \omega) g(\mathbf{k})|^2$ converge to zero for $n \rightarrow \infty$ and all $k \in \mathbb{R}^4$. Furthermore, there is a uniform bound for these functions, namely $|\chi_{\{-1 \leq k^0 - \omega \leq 1\}}(k) g(\mathbf{k})|^2$ (χ_A being the indicator function of the set A) and this uniform bound is integrable w.r.t. the measure $\int_0^\infty \delta_s^+(k) \rho(ds^2)$. Therefore, the right hand side of Eq.(15) converges to zero by the theorem of dominated convergence and we have derived a contradiction. This establishes Theorem 1.

One can summarize the above proof as follows: On shell singularities $\sim 1/\kappa$ in the first argument of a vacuum expectation value which lead to decay reactions are continuous distributions w.r.t. a C^1 -norm and can not be dominated by a measure of Källen-Lehmann type which is (locally) continuous w.r.t. a C^0 -norm.

It should also be noted that if one drops the assumption Eq. (5) one can still conclude $\langle \phi^{\text{out}}(x) \Psi_0, \Psi \rangle = 0 \Leftrightarrow \langle \phi^{\text{in}}(x) \Psi_0, \Psi \rangle = 0$ and thus $\mathcal{H}_1^{\text{in}} = \mathcal{H}_1^{\text{out}}$ with $\mathcal{H}_1^{\text{in/out}} = \overline{\{\phi^{\text{in/out}}(f) \Psi_0 : f \in \mathcal{S}\}}$.

4 A QFT with indefinite metric and decaying particles

Consider the sequence of truncated Wightman functions in energy-momentum variables given by the standard two point functions ($\alpha_l = 1, 2$)

$$\begin{aligned} \left\langle \Psi_0, \hat{\phi}_{\alpha_1}(k_1) \hat{\phi}_{\alpha_2}(k_2) \Psi_0 \right\rangle^T &= \delta_{\alpha_1, \alpha_2} \delta_{m_{\alpha_1}}^-(k_1) \\ &\times \delta(k_1 + k_2) \end{aligned} \quad (16)$$

($\hat{\phi}_\alpha(k)$ being defined as the Fourier transform of the quantum fields $\phi_\alpha(x)$, $\alpha = 1, 2$) and $m_1 = \mu$ and $m_2 = m > 2\mu$ and three point functions

$$\begin{aligned} &\left\langle \Psi_0, \hat{\phi}_{\alpha_1}(k_1) \hat{\phi}_{\alpha_2}(k_2) \hat{\phi}_{\alpha_3}(k_3) \Psi_0 \right\rangle^T \quad (17) \\ &= \prod_{l=1}^3 p_{\alpha_l}(k_l) \sum_{\substack{\beta_1, \beta_2, \beta_3=1,2 \\ j=1,2,3}} \prod_{l=1}^{j-1} \delta_{m_{\beta_l}}^-(k_l) \\ &\times \frac{1}{k_j^2 - m_{\beta_j}^2} \prod_{l=j+1}^3 \delta_{m_{\beta_l}}^+(k_l) \delta(k_1 + k_2 + k_3) \end{aligned} \quad (18)$$

and all other truncated n -point functions equal to zero. Here $p_1(k) = (k^2 - m^2)/(m^2 - \mu^2)$, $p_2(k) = (k^2 - \mu^2)/(m^2 - \mu^2)$. One can prove that (i) the so-defined Wightman functions fulfill the requirements of temperedness, Poincaré invariance, locality, Hermiticity, clustering and spectrality from the Wightman axioms[12], see [2, 6, 7]; (ii) they are the vacuum expectation values of a QFT with indefinite metric in the sense of [9], see [3, 6]; (iii) asymptotic fields for this theory can be constructed and the LSZ asymptotic condition holds in the form of Yang-Feldman equations, cf. [1, 6]; the transition amplitude for the reaction $m \rightarrow \mu + \mu$ for this model is given by

$$2\pi i \delta_m^+(k_1) \delta_\mu^+(k_2) \delta_\mu^+(k_3) \delta(k_1 - (k_2 + k_3)). \quad (19)$$

cf. [1, 6]. Thus, all requirements of Section 2 and 3 with the exception of positivity are not in contradiction with the decay reaction (1). This shows that positivity is really crucial for Theorem 1 and that the energy-time uncertainty relation can be violated in the case of QFTs with indefinite metric.

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